MATH 504 HOMEWORK 7

Due Friday, April 23.

Problem 1. *Suppose that $\mathbb{P}, \mathbb{Q}$ are two posets in the ground model V and $\pi : \mathbb{P} \to \mathbb{Q}$ is a projection, i.e.*

(1) *π is order preserving: $p' \leq p \to \pi(p') \leq \pi(p)$,*
(2) *for all $p \in \mathbb{P}$ and $q \leq \pi(p)$, there is $p' \leq p$, such that $\pi(p') \leq q$.*

Suppose that G is a $\mathbb{P}$-generic filer over V. Show that $H := \{q \mid \exists p \in G, \pi(p) \leq q\}$ is a $\mathbb{Q}$-generic filter over V.

Problem 2. *Suppose that $\mathbb{P}$ is a poset, such that for every $p \in \mathbb{P}$, there are incompatible conditions $q, r \leq p$. Let G be $\mathbb{P}$-generic. Show that $G \times G$ is not $\mathbb{P} \times \mathbb{P}$-generic.*

Problem 3. *Let κ be a regular uncountable cardinal and $\mathbb{P}$ be a κ-closed poset. Show that $\mathbb{P}$ preserves stationary subsets of κ, i.e. if $S \subset \kappa$ is stationary in the ground model, then S remains stationary in any $\mathbb{P}$-generic extension.*
Hint: Given S, a name $\dot{C}$, and p, such that $p \Vdash$ "$\dot{C}$ is a club subset of κ", show there is a sequence in the ground model $\langle p_\alpha, \gamma_\alpha \mid \alpha < \kappa \rangle$, such that:

- *$\langle p_\alpha \mid \alpha < \kappa \rangle$ is a decreasing sequence below p,*
- *$\langle \gamma_\alpha \mid \alpha < \kappa \rangle$ is a club in κ,*
- *each $p_\alpha \Vdash \gamma_\alpha \in \dot{C}$.*

Then use stationarity of S in the ground model.

Problem 4. *Let $S \subset \omega_1$ be a stationary set. Define $\mathbb{P} := \{p \subset S \mid p \text{ is closed and bounded}\}$, and set $p \leq q$ if p end extends q i.e. for some α, $p \cap \alpha = q$. Suppose that $T := S \setminus \omega_1$ is also stationary. Let G be a $\mathbb{P}$-generic filter. Show that in $V[G]$, T is nonstationary.*

The above is an example of a forcing that destroys a stationary set, without collapsing cardinals (we will show that in class, using that S is stationary). On the other hand you cannot destroy a club set:

Problem 5. *If $V \subset W$ are two models of set theory and $V \models$ "D is club in κ", then $W \models$ "D is club in κ".*